The standard notion of approximayte majority refers to distinguishing between n-bit strings that have at least $2n/3$ one-entries and n-bit strings that have at most $n/3$ one-entries. For sake of clarity and generality, we consider the following notion of approximating the Hamming weight of strings.

For $x = x_1 \cdots x_n \in \{0,1\}^n$, we let $\mathrm{wt}(x) \stackrel{\rm def}{=} |\{i \in [n] : x_i = 1\}|$ denote the Hamming weight of x. The relative weight of a string x is $\overline{\mathrm{wt}}(x) \stackrel{\rm def}{=} \mathrm{wt}(x)/|x|$. For fixed $\rho, \epsilon \in (0,1)$, we consider the promise problem in which the YES-instances are strings of relative weight at least ρ and the NO-instances are strings of relative weight at most $\rho - \epsilon$.

We first observe that, for every $x \in \{0,1\}^n$, if we select a random ℓ-subset, denoted $S \subset [n]$, where $\ell = \Theta(\log n)$, then, with probability at least $1 - n^{-3}$, it holds that $\overline{\mathrm{wt}}(x_S) = \overline{\mathrm{wt}}(x) \pm \epsilon/3$, where x_S denotes the projection of x on S.

Selecting $m = n^2$ such subsets $S_1, ..., S_m$, with probability at least $1 - 2^{-n}$, it holds that, for every $x \in \{0,1\}^n$, more than $m - n$ of the S_i's satisfy $\overline{\mathrm{wt}}(x_{S_i}) = \overline{\mathrm{wt}}(x) \pm \epsilon/3$. This is the case because for every $x \in \{0,1\}^n$ we have

$$\begin{aligned}
&\mathrm{Pr}_{S_1,...,S_m \in \binom{[n]}{\ell}} \left[\left| \{ i \in [m] : |\overline{\mathrm{wt}}(x_{S_i}) - \overline{\mathrm{wt}}(x)| > \epsilon/3 \} \right| \geq n \right] \\
&= \binom{m}{n} \cdot \mathrm{Pr}_{S \in \binom{[n]}{\ell}} \left[|\overline{\mathrm{wt}}(x_S) - \overline{\mathrm{wt}}(x)| > \epsilon/3 \} \right]^n \\
&< m^n \cdot (1/n^3)^n \\
&= n^{-n}.
\end{aligned}$$

Hence, there exists a sequence of m sets, denoted $S_1, ..., S_m$, each of size ℓ, such that for every $x \in \{0,1\}^n$ it holds that

$$\left| \{ i \in [m] : |\overline{\mathrm{wt}}(x_{S_i}) - \overline{\mathrm{wt}}(x)| > \epsilon/3 \} \right| < n.$$

Fixing this sequence, and defining $F : \{0,1\}^\ell \to \{0,1\}$ such that $F(z) = 1$ if and only if $\overline{\mathrm{wt}}(z) > \rho - (\epsilon/2)$, we consider the formula

$$\Phi(x) \stackrel{\rm def}{=} \bigvee_{j \in [n]} \bigwedge_{k \in [n]} F(x_{S_{(j-1)n+k}}).$$

Clearly, F can be implemented by a poly(n)-size CNF, and so the forgoing formula is in $\mathcal{AC}^0$. Furthermore, Φ is a monotone formula, because F is a monotone function. We now show that this formula decides correctly on each input that satisfies the promise.

The case of YES-instances: If $\overline{\mathrm{wt}}(x) \geq \rho$, then $\left| \{ i \in [m] : \overline{\mathrm{wt}}(x_{S_i}) \leq \rho - \epsilon/2 \} \right| < n$.

It follows that there exists a $j \in [n]$ such that for every $k \in [n]$ it holds that $F(x_{S_{(j-1)n+k}}) = 1$. Hence, $\Phi(x) = 1$.

The case of NO-instances: If $\overline{\mathrm{wt}}(x) \leq \rho - \epsilon$, then $\left| \{ i \in [m] : \overline{\mathrm{wt}}(x_{S_i}) > \rho - \epsilon/2 \} \right| < n$.

It follows that for every $j \in [n]$ there exists a $k \in [n]$ such that $F(x_{S_{(j-1)n+k}}) = 0$. Hence, $\Phi(x) = 0$.

Perspective: Recall that Majority is not in $\mathcal{AC}^0$. In fact, any constant depth (unbounded fan-in) circuit (with AND, OR, and NOT gates) that computes Majority must have sub-exponential size; specifically, computing n-way Majority in depth d requires size $\exp(\Omega(n^{1/2d}))$.