Elementary Proofs of Set Influence Monotonicity and Sub-Additivity

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For a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, the *influence* of a set of variables $S \subseteq [n]$ on f is defined as

$$I_S(f) \coloneqq \Pr_{x_{[n]\setminus S} = y_{[n]\setminus S}} [f(x) \neq f(y)] .$$

In this note, we prove that influence is monotonic and sub-additive in the set S, using only elementary math.

1 Monotonicity

Claim 1 (Monotonicity). *Let $f\colon \{0,1\}^n \to \{0,1\}$. For any $S \subseteq T \subseteq [n]$, it holds that*

$$I_S(f) \leq I_T(f).$$

Proof. We will first prove the claim for the simpler case when T is a singleton and $S \cup T$ cover all indices. That is, we prove that for any $g\colon \{0,1\}^k \to \{0,1\}$,

$$I_{[k-1]}(g) \leq I_{[k]}(g). \tag{1}$$

First, observe that

$$\begin{aligned} I_{[k]}(g) &= \Pr_{x,y}[g(x) \neq g(y)] \\ &= \frac{1}{2} \cdot \Pr_{x_k = y_k}[g(x) \neq g(y)] + \frac{1}{2} \cdot \Pr_{x_k \neq y_k}[g(x) \neq g(y)] \\ &= \frac{1}{2} \cdot I_{[k-1]}(g) + \frac{1}{2} \cdot \Pr_{x_k \neq y_k}[g(x) \neq g(y)]. \end{aligned}$$

Let x' and y' be distributed uniformly in $\{0,1\}^{k-1}$, and x_k and y_k be sampled from $\{0,1\}$. We ought to show that

$$I_{[k-1]}(g) \leq \Pr_{x',y',x_k \neq y_k}[g(x'x_k) \neq g(y'y_k)]. \tag{2}$$

Denote $p_0 \coloneqq \Pr_{x'}[g(x'0) = 0]$ and $p_1 \coloneqq \Pr_{x'}[g(x'1) = 0]$. Then the left hand side of eq. (2) is

$$\begin{aligned} I_{[k-1]}(g) &= \frac{1}{2} \cdot \Pr_{x',y'}[g(x'0) \neq g(y'0)] + \frac{1}{2} \cdot \Pr_{x',y'}[g(x'1) \neq g(y'1)] \\ &= \frac{1}{2} \cdot (p_0 \cdot (1-p_0) + (1-p_0) \cdot p_0) + \frac{1}{2} \cdot (p_1 \cdot (1-p_1) + (1-p_1) \cdot p_1) \\ &= p_0 \cdot (1-p_0) + p_1 \cdot (1-p_1). \end{aligned}$$

On the other hand, the right hand side of Equation (2) is

$$\Pr_{x',y'}[g(x'0) \neq g(y'1)] = p_0 \cdot (1-p_1) + (1-p_0) \cdot p_1$$

Lastly, note that

$$p_0 \cdot (1-p_0) + p_1 \cdot (1-p_1) \leq p_0 \cdot (1-p_1) + (1-p_0) \cdot p_1,$$

because it is equivalent to $0 \le (p_0 - p_1)^2$. This proves the claim for the simpler case when T is a singleton and $S \cup T$ cover all indices.

Next, we reduce the general case to this simpler case. First, T can be assumed to be a singleton by using induction on $|T$.[1] We can assume without loss of generality that $S = [k-1]$ and that $T = \{k\}$ for some $k < n$. We want to show that

$$I_{[k-1]}(f) \le I_{[k]}(f).$$

Note that the only difference between this case and the simpler case in Equation (2) is that f is defined over a larger domain $[n] \supset [k]$.

We use the notation $x_{[\ell,n]} \coloneqq (x_\ell, \dots, x_n)$. Recall that

$$\begin{aligned} I_{[k]}(f) &= \Pr_{x_{[k+1,n]}=y_{[k+1,n]}}[f(x) \neq f(y)] \\ &= \frac{1}{2} \cdot \Pr_{x_{[k,n]}=y_{[k,n]}}[f(x) \neq f(y)] + \frac{1}{2} \cdot \Pr_{\substack{x_k \neq y_k \\ x_{[k+1,n]}=y_{[k+1,n]}}}[f(x) \neq f(y)] \\ &= \frac{1}{2} \cdot I_{[k-1]}(f) + \frac{1}{2} \cdot \Pr_{\substack{x_k \neq y_k \\ x_{[k+1,n]}=y_{[k+1,n]}}}[f(x) \neq f(y)]. \end{aligned}$$

Therefore, we want to show that

$$\Pr_{x_{[k,n]}=y_{[k,n]}}[f(x) \neq f(y)] \le \Pr_{\substack{x_k \neq y_k \\ x_{[k+1,n]}=y_{[k+1,n]}}}[f(x) \neq f(y)]. \tag{3}$$

Let us be more explicit about how x and y are sampled: to sample a random x and y subject to $x_{[k,n]} = y_{[k,n]}$, one can first sample a shared suffix $x_k v = y_k v$ where $x_k = y_k \in \{0,1\}$ and $v \in \{0,1\}^{n-k}$, then sample prefixes $x', y' \in \{0,1\}^{k-1}$, and finally let $x \coloneqq x' x_k v$ and $y \coloneqq y' y_k v$. Thus, we can rewrite eq. (3) as

$$\Pr_{x',y',x_k=y_k,v}[f(x'x_k v) \neq f(y' y_k v)] \le \Pr_{x',y',x_k \neq y_k,v}[f(x' x_k v) \neq f(y' y_k v)], \tag{4}$$

We claim that Equation (4) follows from the simpler case, or rather, from Equation (2) that was shown therein. In fact, we will show Equation (4) holds "pointwise" in v, that is, that for any $v \in \{0,1\}^{n-k-1}$,

$$\Pr_{x',y',x_k=y_k}[f(x'x_k v) \neq f(y' y_k v)] \le \Pr_{x',y',x_k \neq y_k}[f(x' x_k v) \neq f(y' y_k v)]. \tag{5}$$

Indeed, fix $v \in \{0,1\}^{n-k-1}$, and define $g \colon \{0,1\}^k \to \{0,1\}$ such that $g(x'x_k) \coloneqq f(x' x_k v)$. Then, Equation (5) and Equation (2) are the same, because, on their left hand sides

$$\Pr_{x',y',x_k=y_k}[f(x'x_k v) \neq f(y' y_k v)] = \Pr_{x',y',x_k=y_k}[g(x'x_k) \neq g(y'y_k)] = I_{[k-1]}(g),$$

and on their right hand sides

$$\Pr_{x',y',x_k \neq y_k}[f(x'x_k v) \neq f(y' y_k v)] = \Pr_{x',y',x_k \neq y_k}[g(x'x_k) \neq g(y'y_k)].$$

□

2 Sub-additivity

Claim 2 (Sub-additivity)**.** *Let $f \colon \{0,1\}^n \to \{0,1\}$. For any $S, T \subseteq [n]$, it holds that*

$$I_{S \cup T}(f) \le I_S(f) + I_T(f). \tag{6}$$

[1] That is, if $T = \{i_1, i_2, \dots\}$ and the claim was known for singletons, we would have $I_S(f) \le I_{S \cup \{i_1\}}(f) \le I_{S \cup \{i_2\}}(f), \dots$.

Proof. It will be more illustrative to consider an equivalent definition of set influence. For a set $S \subseteq [n]$, let $V_S \subseteq \{0,1\}^n$ denote the subspace spanned by $\{e_i\}_{i \in S}$. Then,

$$I_S(f) = \Pr_{\substack{x \in \{0,1\}^n \\ v \in V_S}} \left[f(x) \neq f(x+v) \right].$$

Examining the right hand side of eq. (6),

$$\begin{aligned}
I_S(f) + I_T(f) &= \Pr_{\substack{x \in \{0,1\}^n \\ v \in V_S}} [f(x) \neq f(x+v)] + \Pr_{\substack{x \in \{0,1\}^n \\ u \in V_T}} [f(x) \neq f(x+u)] \\
&\geq \Pr_{\substack{x \in \{0,1\}^n \\ v \in V_S, u \in V_T}} [f(x) \neq f(x+v) \vee f(x) \neq f(x+u)] \\
&= 1 - \Pr_{x,v,u} [f(x+v) = f(x) = f(x+u)] \\
&\geq 1 - \Pr_{x,v,u} [f(x+v) = f(x+u)] = \Pr_{x,v,u} [f(x+v) \neq f(x+u)],
\end{aligned}$$

where the first inequality uses the union bound. Substituting $x + v$ with y, we can write

$$\Pr_{\substack{x \in \{0,1\}^n \\ v \in V_S, u \in V_T}} [f(x+v) \neq f(x+u)] = \Pr_{\substack{y \in \{0,1\}^n \\ v \in V_S, u \in V_T}} [f(y) \neq f(x+v+u)] = I_{S \cup T}(f).$$

$\square$